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# INVESTMENT AND DIVIDENDS UNDER IRREVERSIBILITY AND FINANCIAL CONSTRAINTS<sup>▫</sup>

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## Abstract

Research finds that firms' investment and dividend policies are distorted by irreversibility and finance constraints. Whereas the existing literature examines these features separately, this paper considers their interaction. The main theoretical result concerns the separation of the investment and payout thresholds. The ordering of investment and distribution activities is endogenously determined and depends on the levels of capacity and cash balances in a manner consistent with a life-cycle interpretation of firm behaviour. The concavity of the revenue function in capital stock and the complementarity of the constraints drive these results. Important ramifications for empirical work on investment are discussed.

**KEYWORDS:** Investment, dividend policy, irreversibility, financial constraints.

**JEL Classification:** D92, E22, G31, G33, G35.

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## 1 Introduction.

This paper examines the dynamic consequences of irreversibility and financial constraints on a firm's investment and dividend policies. The consensus view is that the frictionless neoclassical model of investment behaviour is flawed. Recent research focuses on irreversibility and financial constraints to explain observed investment behaviour, see Caballero (2000), and Hubbard (1998) for surveys. However, each body of work analyses the impact of one constraint in isolation. Once financial constraints are admitted, in contrast to the results of Modigliani and Miller (1961), dividend and investment activity are both circumscribed and interdependent. This motivates the paper's focus on the interaction of investment, retention and distribution decisions.

Virtually all the existing literature on investment, both empirical and theoretical, considers the impact of one such constraint in isolation.<sup>1</sup> This is problematic because each constraint is used to explain similar features of investment activity,<sup>2</sup> yet the causes of each and hence policy recommendations to which each leads are substantially different. Lack of a theoretical framework which encompasses jointly constrained behaviour has led to an absence of testable implications, while failure even to control for "the other" constraint in empirical work calls into question i) the ability to ascribe investment responses to particular causes, ii) the magnitudes of the estimated effects, iii) the policy conclusions which they generate. Our ability to gain insight is likely to remain compromised until we understand firm behaviour when both constraints may bind simultaneously. Hubbard (1998) (p221-222) argues that steps should be taken to integrate these separate literatures by analysing the impact of both frictions using a real options framework, and by devising empirical tests "to discriminate between the ... two classes of models". In this paper I address both these issues. I use a real options framework to examine the dynamic consequences of the coexistence

<sup>1</sup> Two recent empirical studies formally control for the interaction of the two constraints: Scaramozzino (1997) shows, for a panel of UK firms, that q-theory holds only for the subset of firms for which both irreversibility and financial constraints are unlikely to be present. Guiso and Parigi (1999), using cross-section data on Italian manufacturing firms, find that the impact of uncertainty on investment, is consistent with irreversibility, controlling for the existence of financial constraints.

<sup>2</sup> Irreversibility and financial constraints may each account for several empirical regularities: the history dependence of investment decisions; the existence of hurdle rates for investment and periods of inactivity (threshold effects and nonlinearities); the dominance of quantity variables over price variables in investment equations; the volatility of aggregate investment (both why it is so high and why it is lower than that predicted by a frictionless model).

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of irreversibility and financial constraints on a firm's investment decisions, outline and critically appraise testable restrictions in the light of the existing separate literatures.

Modigliani and Miller (1961) show that the level of dividends does not matter when the firm can raise finance from external sources without incurring a premium over and above internal sources, yet once financial constraints are admitted, dividend policy does affect the value of the firm and investment and dividend policy are not independent, since earnings committed to distribution can not be used for other activities such as investment. As the main focus of the literature has been to identify the rationale for and determinants of dividends, rather than the dynamic interaction with investment activity, the modelling framework for dividend policy has not generally embraced the intertemporal optimising framework of the investment literature (used in this paper).

Often in theoretical work financial constraints on investment or policy arise through informational asymmetries due to selection or hidden action problems, Hubbard (1998). Nonetheless, empirical tests of the implications of this literature adopt an intertemporal neoclassical framework, in which financial constraints are imposed exogenously. Equally, in the irreversible investment literature informational asymmetry (in the form of a lemons problem) is cited as a source of the secondary market imperfection, yet to clarify the dynamic implications (of secondary market imperfections) an intertemporal neoclassical framework is used with an exogenously imposed irreversibility constraint, Dixit and Pindyck (1994). This approach is adopted below, hence the paper can be interpreted as an extension of the canonical irreversible investment model.

The irreversibility constraint is standard. Financial constraints, following Milne and Robertson (1996) are captured through the unavailability of external funds combined with the threat of liquidation should internal funds fall too low. By incorporating irreversible investment this paper adds greater realism to the threat of liquidation. In Milne and Robertson's model, symmetric convex costs of investment are adopted. This allows a firm to manage its physical capital at limited cost in the face of an increased threat of liquidation due to lower internal funds. Once investment is irreversible, such behaviour not feasible. Thus a key difference between the current paper and that of Milne and Robertson (1996) is that the irreversibility constraint acts to complement the financial constraint. This complementarity is likely to be robust provided investment is subject to

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non-convex adjustment technology, such as the costly reversibility discussed in Abel and Eberly (1996). This is because the capital loss involved in selling equipment to the secondary market will be more severe under non-convexities than under symmetric convex costs (considered by Milne and Robertson) and will limit scope for managing capital to offset movements in internal funds.

One of the contributions of the paper lies in the analytic characterisation of the investment and dividend thresholds. It is shown that investment and dividend thresholds are separate and are nonlinear in capital stock. This is a direct consequence of the concavity of the revenue function in capital stock, in conjunction with the constraint complementarity. These results seem to be robust to alternative operating environments precisely because the revenue concavity and constraint complementarity properties are likely to be preserved. The second main contribution of the paper lies in documenting the consequences for empirical work on investment of the juxtaposition of irreversibility and financial constraints. This discussion highlights the importance of controlling for one constraint when examining the impact of the other. In particular the impact of the complementarity of these constraints suggests that it is better to work with data where one constraint is unlikely to be present.

The next section outlines a model of firm behaviour under irreversibility and financial constraints. In the following section, properties of the firm's value function and the optimal policy are investigated. Implications for empirical work are discussed in Section (4). The robustness of the results to a series of extensions of the firm's operating environment is discussed in Section (5). Section (6) contains a conclusion and some suggestions for future research.

## 2 The Model.

This section sets out a simple framework for the analysis of investment and dividend policy under irreversibility and financial constraints. The starting point for the analysis is the sources and uses of funds identity. A simple continuous time version of this relation may be written as

$$\frac{1}{4}(t) dt + dS(t) + dB(t) = dX(t) + (P(t)I(t) + D(t))dt: \quad (1)$$

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In words, over any time interval,  $dt$ , net profits from operations,  $\frac{1}{4}(t)$ , plus the net issue of equity,  $dS(t)$ , plus the net issue of debt,  $dB(t)$ , (which together comprise the sources of funds), are identically equal to the change in retained earnings,  $dX(t)$ , plus expenditure on investment,  $P(t)I(t)$ , and dividends paid out,  $D(t)$  (these comprise the uses of funds).

Let net earnings from operations be  $\frac{1}{4}(t) = P(t)Q(t) - K(t)$ , where  $Q(t)$  is output and  $P(t)$  is output price. Assume that the firm's production function is continuous and concave in a single factor, capital equipment, in particular let  $Q(K) = K^\alpha$ . If this set up is adopted in relation to the sources and uses of funds identity, then even if  $P$  is assumed to be time invariant, there will be five state variables:  $B; K; P; S; X$  and two controls  $D, I$ . Such a structure is too complex to facilitate analytic results concerning the firm's optimal policies. Rather than obtain numeric results concerning special cases, I use a series of simplifying assumptions to obtain analytic results. I consider the implications of relaxing some of these assumptions in Section (5).

First set output price,  $P$ , and the price of capital equipment,  $P_K$ , as constant  $\$1$ , and normalise the former to unity. Normalising  $P$  to unity simplifies the problem somewhat, but it also omits the prime source of uncertainty in canonical irreversible investment models, which makes evaluation of the results more difficult. Instead uncertainty is allowed to enter profits directly. The flow of net profits over the time interval  $dt$  is then

$$\frac{1}{4}(t) dt = K(t)^\alpha dt + \frac{1}{2} K(t)^\alpha dW(t); \quad (2)$$

where  $W(t)$  is a Wiener process.

Irreversible investment is frequently motivated by appealing both to asset specificity and to "lemons" problems in the secondary market for capital equipment. This is used to justify an (exogenous) irreversibility constraint on investment in equipment: researchers do not feel compelled to model the informational asymmetry directly, but instead impose the constraint and evaluate its consequences. Similarly, financial constraints are often seen as the consequence of informational asymmetries between the firm and the providers of external finance. Here I impose an exogenous financial constraint, and proceed to evaluate its behavioural consequences.

In particular, financial constraints can be incorporated into the model by assuming that the firm

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...nds it prohibitively expensive to raise external ...nance. This can occur as a consequence of imperfect knowledge amongst prospective equity and debt holders of the current state of and prospects for the ...rm. For instance Hellmann and Stiglitz (2000) outline conditions under which a ...rm may be credit and equity rationed simultaneously under adverse selection. To be precise this assumption restricts new issues of equity and new borrowing. It does not affect repurchases of equity or redemption of outstanding debt. To simplify the analysis it is assumed that repurchasing and redemption can not occur; these constraints involve the restrictions  $dB(t) = dS(t) = 0$ ; <sup>3</sup> Under these assumptions, the sources and uses of funds identity may be rearranged to give the evolution of cash balances as<sup>3</sup>

$$dX(t) = (K(t)^\circ - P - I(t) - D(t))dt + \frac{1}{2}K(t)^\circ dW(t) : \quad (3)$$

To allow ...nancial constraints to bite, assume that the ...rm faces automatic liquidation if internal cash balances,  $X(t)$ , fall below some threshold. This makes internal sources of funds a hedge against the threat of liquidation. For simplicity, the threshold is set at zero, which is consistent with irreversibility of investment, and gives the restriction<sup>4</sup>

$$X(t) \geq 0 : \quad (4)$$

The concept of the irreversibility of dividend payments is captured through the constraint,

$$D(t) \geq 0 : \quad (5)$$

Irreversibility of investment is captured by the standard constraint,

$$I(t) \geq 0 : \quad (6)$$

The result is that the level of capital stocks can only fall if depreciation is positive. Suppose, for convenience that the rate of depreciation is zero. Then capital stock only changes when investment is undertaken, so that the capital accumulation equation becomes

$$dK(t) = I(t)dt : \quad (7)$$

<sup>3</sup> For simplicity it is assumed that no interest can be earned on retained earnings.

<sup>4</sup> Under these assumptions, since capital has no resale value, the net worth of the ...rm equals its cash balance,  $X$ .

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The analytical advantage of this simplification is that while capital remains a state variable, so that capacity choice matters, the dimension of the differential equation governing  $V$  in the continuation region is reduced by one.<sup>5</sup> In other words, capital stock acts as a scaling variable. This also simplifies the analysis of the regions of state space in which the various regimes (inaction, investment, etc.) hold.

The firm's objective is to maximise the discounted value of a stream of dividend payments,  $D$ , by judicious control of dividend and investment policies. Its value function is given as

$$V(K(t); X(t)) = \max_{D(s); I(s)} E_t \int_t^{\infty} e^{-\frac{1}{2}(s-t)} D(s) ds + e^{-\frac{1}{2}(\ell-t)} V(K(\ell); 0); \quad (8)$$

subject to (3), (4), (5), (6) and (7). Here  $E_t$  denotes the expectation operator conditional on the information available at time  $t$ ,  $\ell$  is the first time that cash balances fall to zero, so that the firm enters liquidation and  $V(K(\ell); 0) = 0$  by assumption. Note  $X(t) = 0$  forms an absorbing state.

The firm's behaviour is an optimal resolution of the conflicts between the desire to pay irreversible dividends to impatient shareholders, the requirement to retain cash as a barrier against liquidation, and the desire to boost future profits by undertaking irreversible investment expenditures to increase future revenues. The impatience of the firm's owners is captured by the assumption that shareholders rate of time preference,  $\frac{1}{2}$ , is strictly non-negative,  $\frac{1}{2} > 0$ . If this condition did not hold then the firm would maximise value by accumulating capital and cash balances into the indefinite future so as to render the threat of liquidation inoperative.

### 3 Solution to the Firm's Problem.

The solution to the firm's problem is considered in this section. This involves identifying the form of the solution outside and inside the continuation region and determining two sets of unknowns: constants of integration and the location of the action thresholds (the latter defines the shape of the continuation region). The analysis is complicated by the existence of several relevant unknown action thresholds associated with payout, investment and joint action regimes and by the complementarity of irreversibility and financial constraints. The approach used is first to determine the Hamilton-Jacobi-Bellman (HJB) equation governing  $V$ . Then the payout and

<sup>5</sup> Pindyck (1988) adopts such a procedure to analyse the investment decision of a financially unconstrained firm.



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investment thresholds are characterised under restrictions on the curvature of the value function. Finally, value function itself is identified, conditional on the thresholds.

### 3.1 The Hamilton-Jacobi-Bellman Equation.

The HJB equation satisfied by the firm's value function is obtained by the standard approach of expanding the integral (8) over an interval  $[t; t + dt]$  using the evolution equations (3) and (7), taking the limit as  $dt \rightarrow 0$  using Ito's lemma, and dividing by  $dt$  to obtain

$$\frac{1}{2} V(K(t); X(t)) = \max_{I, D} \left\{ \frac{\sigma^2 K(t)^2}{2} V_{XX}(K(t); X(t)) + K(t) V_X(K(t); X(t)) + (P - I(t) + D(t)) V_X(K(t); X(t)) - I(t) V_K(K(t); X(t)) + D(t) V_D(K(t); X(t)) \right\} \quad (9)$$

To proceed it is necessary to undertake the maximisation with respect to  $I$  and  $D$  in order to determine the optimal policy rules for these variables. In addition it is important to understand conditions under which the optimal policies are feasible. These issues are discussed next.

### 3.2 The Structure of Optimal Dividend and Investment Policy.

In this sub-section it is shown that, under restrictions on the value function given in Conjecture (1), the investment and payout thresholds are bounded and non-negative.

**Conjecture 1** Suppose that the value function  $V(K; X)$  is twice continuously differentiable with derivatives  $V_K > 0$ ;  $V_{KK} < 0$ ,  $V_X > 0$ ,  $V_{XX} < 0$  and  $V_X(K; 0) > 1$ ,  $V_K(0; X) > P$ .

These conditions are reasonable.  $V_K > 0$ ;  $V_{KK} < 0$  and  $V_K(0; X) > P$  are satisfied for irreversible investment under perfect capital markets, while  $V_X > 0$ ,  $V_{XX} < 0$  and  $V_X(K; 0) > 1$  hold in Milne and Robertson's model.<sup>6</sup> Under Conjecture (1) the structure of the optimal policies is summarised in the following statement:<sup>7</sup>

**Proposition 2** Consider a firm with value function (8) subject to the evolution equations (3) and (7) and constraints (4), (5) and (6). Assume that the value function satisfies Conjecture (1), then a) Conditioning on the level of capital stock, the optimal dividend policy takes the form:  $\exists X_D(K)$  s.t.  $D = 0$ ;  $\forall X > X_D(K)$ , with instantaneous barrier control at  $X = X_D(K)$ , where  $X_D(K) \geq 0$ .

<sup>6</sup> However, it is necessary to verify the statement. A direct analytic proof is not feasible, since the value function must be computed numerically. Nonetheless it is a simple matter to verify numerically that the solution to the firm's problem outlined in this section has these properties throughout the inaction region for various sets of parameter values (on a finite grid in  $(K; X)$ -space).

<sup>7</sup> This proposition differs from Milne and Robertson (1996) in that statement a) generalises Milne and Robertson's statement to allow for dependence of the dividend threshold on  $K$ ,  $\partial X_D / \partial K > 0$ , not just a subset of possible values. Statement b) has no counterpart in Milne and Robertson (1996), since they do not consider irreversible investment.

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$(0; 1)$ . b) Optimal investment for given capacity take the form  $9K_I$  s.t.  $I = 0; 8K < K_I$ , with instantaneous barrier control at  $K = K_I$ ; where  $K_I \in (0; 1)$ .

Proof. See Appendix. ■

This result indicates that the optimal investment threshold bounds  $K$  from below while the payout threshold bounds  $X$  from above. Since  $K$  is fixed in the absence of investment activity, it is simpler, from a notational viewpoint, to write the investment threshold as  $X_I \leftarrow X_I(K)$ . The optimal policy is that the firm invests when  $X > X_I(K)$ . Proposition (2) indicates that state space is divided into a number of regions which firms find it optimal to do one of a number of things: i) do nothing ( $X < X_D(K), X < X_I(K)$ ), ii) invest ( $X < X_D(K), X \geq X_I(K)$ ), iii) payout ( $X \geq X_D(K), X < X_I(K)$ ), iv) payout and invest ( $X \geq X_D(K), X \geq X_I(K)$ ), v) enter liquidation  $X = 0$ .<sup>8</sup> Solution of the firm's problem involves identifying the regions of state space where these regimes hold and determining the rules governing transitions between regimes; so the optimisation problem determines an optimal hierarchy of actions. The novel feature, and the key reason that the threshold rules and the optimisation problem differ from those in the canonical irreversible investment model, Bertola (1998) and Milne and Robertson's (1996) threat of liquidation model is that the irreversibility and financial constraints complement each other and hence investment and dividend policy, are interdependent.<sup>9</sup> The full solution requires boundary conditions, which are discussed in the next sub-section.

### 3.3 Optimal Policies, Action Thresholds and Boundary Conditions.

Further progress towards understanding the firm's behaviour requires that the constraints under which it operates be mapped into boundary conditions at the edge of the continuation region.<sup>10</sup>

There are two classes of boundary fixed and free, these are analysed in turn.

Fixed boundaries, whose locations are known, arise from the existence conditions for the firm.

At liquidation  $X = 0$ , then  $V(K; 0) = 0, 8K > 0$ , since capital has no resale value. This is an

<sup>8</sup> In the empirical literature examining investment under financial constraints, dividend policy is sometimes used as a means of classifying financially constrained firms (low or zero dividend firms are seen as financially constrained). In the context of the model in this paper, any firm paying dividends is, de facto, financially unconstrained, while firms are financially constrained to the extent that the ratio of  $X=K$  falls short of that which triggers dividend payouts at that scale of operation.

<sup>9</sup> If the constraints were independent, then the  $X_D(K)$  would be horizontal in  $(K; X)$ -space, whilst  $X_I(K)$  would be vertical. The extent of the interdependence is displayed in Figure (3), and will be explained in Section (4).

<sup>10</sup> Once the firm is in the continuation region, regulation (investment and/or dividend payment) prevents it from leaving except through liquidation. For the present assume that the initial  $(K; X)$  pair lies inside the continuation region. The situation where the initial  $(K; X)$  lie outside the continuation region is dealt with in Section (3.7).

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absorbing state. Also, since a firm with neither physical capital nor cash balances is of no value to its owners, the required condition at  $K = X = 0$  is  $V(0; 0) = 0$ . These conditions are summarised as

$$\begin{aligned} V(K; 0) &= 0; \quad \forall K > 0; \\ V(0; 0) &= 0; \end{aligned} \tag{10}$$

The boundary conditions at the free boundaries indicate when and where it becomes optimal to undertake investment and distribution (rather than retention) under the behavioural constraints, (5) and (6). Along the payout action threshold the condition is  $V_X(K; X) = 1$  and along the investment threshold  $V_K(K; X) = P V_X(K; X)$ . These (necessary) 'smooth-pasting' conditions follow from the sample path continuity of Brownian motion, but are insufficient to determine the optimal policies as they do not identify the location of the action thresholds. Sufficient conditions are given by the second derivatives of the value function along the respective thresholds:<sup>11</sup>

$$\begin{aligned} V_X &= 1 & \text{at } X = X_D(K); \\ V_{XX} &= 0 & \text{at } X = X_D(K); \end{aligned} \tag{11}$$

$$\begin{aligned} V_K &= P V_X & \text{at } X = X_I(K); \\ V_{XK} &= P V_{XX} & \text{at } X = X_I(K); \end{aligned} \tag{12}$$

It is worth pausing to interpret these conditions. Take the payout decision. For  $0 < X < X_D$ , an extra unit of operating profit is worth more to the firm held as cash balances,  $X$ , to stave off the threat of liquidation, than distributed as dividends to the owners. That is  $V_X(K; X) > 1$ . When dividends are paid the marginal value of cash balances has fallen to the marginal value of dividends,  $V_X(K; X) = 1$ . Next, consider the investment decision. Recall that marginal  $q$  is given as the ratio of the marginal value product of capital to the replacement cost of capital,  $q = \frac{V_K}{P}$ . In the standard irreversible investment model, the firm invests when  $q$  rises above unity.<sup>12</sup> The firm invests when  $q$  rises to the marginal value of cash balances,  $q = V_X(K; X)$ . When  $0 < X < X_I$ , the firm does not invest since its marginal valuation of the extra unit of capital is less than the marginal unit of capital's value at replacement cost, weighted by the marginal value of cash

<sup>11</sup>Since the form of such conditions is standard within the literature, Dixit and Pindyck (1994), proofs are omitted.  
<sup>12</sup>This marginal  $q$  based investment rule is the value of the firm form discussed by Dixit and Pindyck (1994). Although this looks much like the rule used in frictionless models of investment, those models neglect the fact that the marginal value product of capital accounts for the effects of exercising the option to invest.

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balances. In other words, it is only worth investing when the capitalised revenue flows associated with the marginal unit of capital exceeds the purchase price of new capital equipment weighted by a factor measuring the value of (detrimental effect of investment on) the incremental unit of cash.

It is quite possible that a firm can choose to pay dividends and undertake investment simultaneously. Then substituting the boundary conditions at the payout threshold, equation (11), into those at the investment threshold, equation (12), three conditions summarise the optimal policy under joint action,

$$\begin{aligned} V_K &= P \\ V_{KX} &= 0; (K; X) \in (K; X) \in R_+^2; X \leq X_I(K) = X_D(K) = X_{ID}(K) \\ V_{XX} &= 0 \end{aligned} \quad (13)$$

Having re-expressed the inequality constraints (4), (5) and (6) as boundary conditions on the firm's value function, the next step is to determine some general properties of the value function.

### 3.4 Some General Properties of the Value Function.

The main result of this section lies in the identification of the form of the firm's value function, subsidiary results determine elements of this function.

The boundary conditions outlined in Section (3.3) may be substituted back into (9), giving a differential equation in  $V$ . To recap: the structure of the firm's problem has been simplified by assuming that capital assets do not depreciate either over time or with use, that cash balances, rather than product demand, is the source of innovations and that no interest can be earned on cash balance. The first two assumptions allow the problem to be modelled in two dimensions by solving a linear homogeneous second order ODE in  $X$ , equation (14), with the other state variable, capital, acting as a scaling factor. The final assumption ensures that this ODE is of constant coefficient form. Over the continuation (inaction) region, for which the constraints (5) and (6) hold as equalities, this differential equation is

$$\frac{\sigma^2 K^2}{2} V_{XX} + K^\alpha V_X - \frac{1}{2} V = 0; \quad (14)$$

subject to the boundary conditions (10), (11) and (12). The general solution of equation (14)

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takes the form

$$V(K; X) = A_1(K) e^{-\lambda_1 X} + A_2(K) e^{-\lambda_2 X}; \quad (15)$$

where  $\lambda_i, i = 1, 2$ , are the roots of the characteristic polynomial

$$\frac{\beta^2}{2} K^{2\beta-2} + K^\beta - \lambda = 0;$$

That is,

$$\lambda_i = \frac{\beta-1}{\beta} \frac{P}{K^\beta}; \quad i = 1, 2; \quad (16)$$

Taking  $\lambda_1$  as the positive root of the characteristic equation, since  $\beta/2; K^\beta \geq 0$ , it follows that  $\lambda_2 > \lambda_1$  and  $\lambda_2 < \frac{\beta-1}{\beta} \frac{P}{K^\beta} < 0 < \lambda_1$ . The characteristic roots can be written as functions of  $K$ ,  $\lambda_i = \lambda_i(K), i = 1, 2$ .

One of the constants of integration,  $A_1(K)$  and  $A_2(K)$ , may be identified using the mixed boundary conditions: from equations (10) and (15), it follows that  $A_1(K) = \lambda_2 A_2(K)$ , thus

$$V(K; X) = A_1(K) e^{-\lambda_1(K)X} + \lambda_1(K) e^{-\lambda_2(K)X}; \quad (17)$$

To determine the solution fully, the constants of integration and locations of the action thresholds must be determined under the different regimes. The action thresholds are identified in Section (3.5); the constants of integration are discussed in Section (3.6).

### 3.5 Identification and Characterisation of the Action Thresholds.

The aim of this section is to express  $X_I$ , and  $X_D$  as functions of  $K$ , and determine the slope and curvature of these functions. This information determines the locations of the different regimes. The behaviour of the investment and payout thresholds are considered separately.

#### 3.5.1 The Payout Threshold.

The payout threshold is obtained by imposing the boundary conditions (11) on the value function (17). A closed form representation is obtained. The results are summarised as follows

**Proposition 3** When the value function of the firm outlined in Section (2) satisfies equation (17) subject to equations (10), (11), (12) and (13), the payout threshold is globally concave in  $K$ ,  $\partial K > 0$ , and is given by

$$X_D(K) = \frac{\beta^2 \ln \frac{1 + (1 + 2\beta/4^2) + 2\beta \frac{P}{1 + 2\beta/4^2}}{1 + (1 + 2\beta/4^2)\lambda_1^2 \frac{P}{1 + 2\beta/4^2}}}{2 \frac{P}{1 + 2\beta/4^2}} K^\beta; \quad (18)$$

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Proof. See Appendix. ■

The curvature of the payout threshold plays a key role in determining the regime (payout or investment) under which the firm operates.

## 3.5.2 The Investment Threshold.

The analysis of the investment threshold is more convoluted than that for the payout threshold because it is only possible to obtain implicit expressions for  $X_I(K)$ . One can solve this numerically, but to determine the location of the investment threshold directly, extra information is used drawing on the (boundary) conditions which hold under joint action. This allows the location of the investment threshold to be determined at two points in  $(K; X)$ -space. The main result of this section is that for given parameter values, joint action occurs only at  $(0; 0)$  and a unique non-negative pair  $(K_{ID}; X_{ID})$ .

First, an implicit function defining the investment threshold is derived. Given an initial level of capacity  $K : X_I(K) < X_D(K)$ , the free boundary,  $X_I(K)$  is determined by imposing the boundary conditions (12) on the value function (17).

**Lemma 4** When the value function of the firm outlined in Section (2) satisfies equation (17) subject to equations (10), (11), (12) and (13), the investment threshold is defined implicitly by

$$G(K; X_I) = \frac{2}{4} \frac{e^{(-1(K) i - 2(K)) X_I} + \frac{e^{(-2(K) i - 1(K)) X_I}}{(-1(K) i - 2(K))} + \frac{e^{(-2(K) i - 1(K)) X_I}}{(-1(K) i - 2(K))^2 P} + \frac{e^{(-2(K) i - 1(K)) X_I}}{(-1(K) i - 2(K))} = 0 \quad (19)$$

Proof. See Appendix. ■

$G(K; X_I)$  represents the points for which the general surface  $G(K; X)$  is equal to 0.<sup>13</sup> Noting the explicit functional form for output  $Q = K^\alpha$ ,  $0 < \alpha < 1$ , write  $G(K; X)$  as

$$G(K; X) = \frac{2}{4} C_1 e^{AXK^i} K^{i(\alpha+1)} + C_2 e^{iAXK^i} K^{i(\alpha+1)} + \frac{2}{5} \frac{A^2 P K^{i2\alpha} + A^2 X K^{i(2\alpha+1)} + C_3 K^{i(\alpha+1)}}{5} \quad (20)$$

where  $C_1 = i^\alpha \frac{P}{1 + 2\frac{1}{2}\frac{1}{4}^2} < 0$ ,  $C_2 = i^\alpha \frac{P}{1 + 2\frac{1}{2}\frac{1}{4}^2} > 0$ ,  $C_3 = \frac{P}{1 + 2\frac{1}{2}\frac{1}{4}^2} < 0$ ,  $A = \frac{P}{1 + 2\frac{1}{2}\frac{1}{4}^2}$ .

Figure (1) illustrates the behaviour of  $G(K; X)$  over a region of state space for  $\frac{1}{2} = 0.1$ ,  $\frac{1}{4} = 0.5$ ,

$\alpha = 0.7$ ,  $P = 1$ . The behaviour of the surface  $G(K; X)$  determines the existence of the investment

<sup>13</sup> The notion of using the boundary conditions to define an implicit function recalls Abel and Eberly's (1996) study of costly reversible investment. However, in that paper the implicit function defined the ratio of the threshold values at acquisition and disposal as a function of primary to secondary market price (the degree of irreversibility). Here the investment threshold  $X_I$  is directly, if implicitly, identified by  $K$ .

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threshold. It was not possible to determine a closed form solution to the equation  $G(K; X) = 0$  for  $0 < X < 1$ . The asymptotic properties of this surface are considered in the Appendix and used to deduce the existence of the investment threshold.

Although, full characterisation of the investment threshold is only feasible numerically, analytical methods may be used to determine the location of the investment threshold at the joint action point. That is the point where the investment and dividend thresholds interact. To analyse this situation, simplify expression (20) by multiplying throughout by  $K^{1+\alpha}$  to give

$$H(K; X_I) = \frac{6}{4} \frac{C_1 e^{\alpha X_I K^{1+\alpha}} + C_2 e^{i \alpha X_I K^{1+\alpha}}}{\hat{A}^2 P K^{1+\alpha} + \alpha \hat{A}^2 X_I K^{1+\alpha} + C_3} = 0: \quad (21)$$

For  $X_I = 0$ , it follows that  $K = 0$ , since

$$C_1 + C_2 + C_3 + \hat{A}^2 P K^{1+\alpha} = \hat{A}^2 P K^{1+\alpha} = 0:$$

Note that the payout threshold when  $K = 0$ , is also zero,  $X_D(K) = 0$  - the payout and investment threshold intersect at  $(K; X) = (0; 0)$ . To understand the firm's optimal policies, it is important to identify regions of state space where  $X_I(K) \leq X_D(K)$ . First, note that equation (18) which defines  $X_D(K)$  appears very different from equation (21) determining  $X_I(K)$ , so the two thresholds are not likely to be superposed. However, they may intersect. Below, it is shown that there is a unique intersection,  $X_D(K) = X_I(K) = X_{ID}(K_{ID})$ , for positive  $K = K_{ID}$ . To see this substitute for  $X_I(K_{ID})$  in equation (21) using equation (18):

$$X_D(K_{ID}) = \frac{C_4}{\hat{A}} K_{ID}^{1+\alpha} \quad (22)$$

where  $C_4 = \ln \frac{1 + \frac{1}{2}\alpha^2 + \frac{P}{1 + 2\frac{1}{2}\alpha^2}}{1 + \frac{1}{2}\alpha^2 i \frac{P}{1 + 2\frac{1}{2}\alpha^2}}$ . Then

$$H(K_{ID}) = \frac{6}{4} \frac{C_1 \frac{1 + \frac{1}{2}\alpha^2 + \frac{P}{1 + 2\frac{1}{2}\alpha^2}}{1 + \frac{1}{2}\alpha^2 i \frac{P}{1 + 2\frac{1}{2}\alpha^2}} + C_2 \frac{1 + \frac{1}{2}\alpha^2 i \frac{P}{1 + 2\frac{1}{2}\alpha^2}}{1 + \frac{1}{2}\alpha^2 + \frac{P}{1 + 2\frac{1}{2}\alpha^2}}}{\hat{A}^2 P K_{ID}^{1+\alpha} + \alpha \hat{A} C_4 + C_3} = 0:$$

This nonlinear equation has two solutions  $K_{ID0} = 0$  and

$$K_{ID1} = \frac{6}{4i} \frac{C_1 \frac{1 + \frac{1}{2}\alpha^2 + \frac{P}{1 + 2\frac{1}{2}\alpha^2}}{1 + \frac{1}{2}\alpha^2 i \frac{P}{1 + 2\frac{1}{2}\alpha^2}} + C_2 \frac{1 + \frac{1}{2}\alpha^2 i \frac{P}{1 + 2\frac{1}{2}\alpha^2}}{1 + \frac{1}{2}\alpha^2 + \frac{P}{1 + 2\frac{1}{2}\alpha^2}} + \alpha \hat{A} C_4 + C_3}{\hat{A}^2 P}^{\frac{1}{1+\alpha}}: \quad (23)$$

It is not possible to determine the sign of this expression analytically, but numerical analysis showed that  $K_{ID1} > 0; 8\frac{1}{2} \leq (0; 0.25), \frac{1}{2} \leq (0; 1)$ : allowing a broad range of parameter values.

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## 3.5.3 Action Threshold Curvature and the Optimal Regime.

This section demonstrates that the optimal policy partitions state space into two regions where different policy regimes are relevant. Investment is the preferred action (whenever action is optimal)  $8K \geq 2(0; K_{ID_1})$ , and payout is the preferred action (whenever action is deemed optimal)  $8K \geq 2(K_{ID_1}; 1)$ , while joint action occurs only at  $K = K_{ID_1}$ . The following result is established by comparing the elasticity of investment threshold and dividend threshold at the joint action point  $(K_{ID_1}; X_{ID_1})$ .

**Proposition 5** For  $K \geq 2(0; K_{ID_1})$ ,  $X_I(K) < X_D(K)$ , that is investment is the preferred action, and for  $K \geq 2(K_{ID_1}; 1)$ ,  $X_D(K) < X_I(K)$ , that is payouts are the preferred action.

**Proof.** Since the dividend and investment thresholds intersect only at  $(K; X) = (0; 0)$  and  $(K; X) = (K_{ID_1}; X_{ID_1})$ , it must be the case that at  $(K_{ID_1}; X_{ID_1})$  either the investment threshold cuts the dividend threshold from below or vice versa. It follows that a change of regime occurs at  $K = K_{ID_1}$ . The relative slopes of the investment and dividend thresholds at  $(K_{ID_1}; X_{ID_1})$  determine which regime dominates for  $K \geq K_{ID_1}$ . These relative slopes are captured in the elasticities of the thresholds with respect to  $K$  at  $K = K_{ID_1}$ .

The elasticity,  $\frac{dX_D(K)}{dK} \frac{K}{X_D(K)}$  of the payout threshold,  $X_D(K)$ , with respect to  $K$  is

$$\eta_D = \frac{dX_D(K)}{dK} \frac{K}{X_D(K)} \bigg|_{(K;X)=(K_{ID_1};X_{ID_1})} = 0; \quad 8K > 0$$

For the investment threshold, apply the implicit function theorem to equation (21) to obtain

$$\frac{dX_I(K)}{dK} = \frac{\frac{\partial}{\partial K} \{ \bar{A}C_1 X_I K^{i-1} e^{AX_I K^{i-1}} + \bar{A}C_2 X_I K^{i-1} e^{AX_I K^{i-1}} + \bar{A}^2 X_I K^{i-1} \bar{A}^2 (1-i) P \}}{\bar{A}C_1 e^{AX_I K^{i-1}} + \bar{A}C_2 e^{AX_I K^{i-1}} + \bar{A}^2} \quad (24)$$

Then, since at the joint action point  $X_I(K_{ID_1}) = X_{ID_1}(K_{ID_1}) = X_D(K_{ID_1})$ , the elasticity of the

of the investment threshold at the joint action point can be written as  $\frac{dX_I(K)}{dK} \frac{K}{X_I(K)} \bigg|_{(K;X)=(K_{ID_1};X_{ID_1})}$

$= \frac{dX_I(K)}{dK} \frac{K}{X_D(K)} \bigg|_{(K;X)=(K_{ID_1};X_{ID_1})}$  "1: So

$$\eta_I = \frac{\frac{\partial}{\partial K} \{ \bar{A}C_1 X_D(K_{ID_1}) K_{ID_1}^{i-1} e^{AX_D(K_{ID_1}) K_{ID_1}^{i-1}} + \bar{A}C_2 X_D(K_{ID_1}) K_{ID_1}^{i-1} e^{AX_D(K_{ID_1}) K_{ID_1}^{i-1}} + \bar{A}^2 X_D(K_{ID_1}) K_{ID_1}^{i-1} \bar{A}^2 (1-i) P \}}{\bar{A}C_1 e^{AX_D(K_{ID_1}) K_{ID_1}^{i-1}} + \bar{A}C_2 e^{AX_D(K_{ID_1}) K_{ID_1}^{i-1}} + \bar{A}^2} \frac{K_{ID_1}}{X_D(K_{ID_1})} \quad (25)$$



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Substituting in equation (25) for  $X_D(K_{ID})$  and  $K_{ID}$  using equations (22) and (23) and rearranging gives

$$\pi_I = \pi^0 + \frac{(1 - \pi^0) \left[ C_1 \frac{1 + \frac{1}{2}\gamma^2 + \frac{p}{1+2\frac{1}{2}\gamma^2}}{1 + \frac{1}{2}\gamma^2} + C_2 \frac{1 + \frac{1}{2}\gamma^2}{1 + \frac{1}{2}\gamma^2 + \frac{p}{1+2\frac{1}{2}\gamma^2}} + \pi^0 \bar{A} C_4 + C_3 \right]}{C_1 e^{C_4} + C_2 e^{C_4} + \pi^0 \bar{A}} \quad (26)$$

So the difference between the elasticities of the investment and payout thresholds at the joint action point is

$$\pi_I - \pi_D = \frac{(1 - \pi^0) \left[ C_1 \frac{1 + \frac{1}{2}\gamma^2 + \frac{p}{1+2\frac{1}{2}\gamma^2}}{1 + \frac{1}{2}\gamma^2} + C_2 \frac{1 + \frac{1}{2}\gamma^2}{1 + \frac{1}{2}\gamma^2 + \frac{p}{1+2\frac{1}{2}\gamma^2}} + \pi^0 \bar{A} C_4 + C_3 \right]}{C_1 e^{C_4} + C_2 e^{C_4} + \pi^0 \bar{A}} \cdot \frac{N}{D}$$

It follows from equation (23) that the numerator,  $N$ , of this expression is negative (at least  $8\frac{1}{2} > 0$ ;  $0.25$ ),  $\frac{3}{4} > 0$ ;  $1$ ). Moreover the denominator is also negative. To see this substitute for  $C_1$ ,  $C_2$  and  $\bar{A}$  and express the denominator,  $D$ , directly as a function of  $\pi^0$ ;  $\frac{1}{2}$ ;  $\frac{3}{4}$ :

$$D = \frac{8}{\pi^0} \left[ \frac{1 + \frac{1}{2}\gamma^2 + \frac{p}{1+2\frac{1}{2}\gamma^2}}{\frac{3}{4}^2} + \frac{1 + \frac{1}{2}\gamma^2}{1 + \frac{1}{2}\gamma^2 + \frac{p}{1+2\frac{1}{2}\gamma^2}} + \frac{9}{\pi^0} \left[ \frac{1 + \frac{1}{2}\gamma^2}{1 + \frac{1}{2}\gamma^2 + \frac{p}{1+2\frac{1}{2}\gamma^2}} + \frac{p}{1 + \frac{1}{2}\gamma^2 + \frac{p}{1+2\frac{1}{2}\gamma^2}} + \frac{2}{\frac{3}{4}^2} \right] \right]$$

Some algebraic manipulation gives the result

$$D = \frac{4}{\pi^0} \frac{1 + 2\frac{1}{2}\gamma^2}{\frac{3}{4}^2} = \frac{2}{\pi^0} \bar{A} < 0:$$

This establishes that  $\pi_I - \pi_D > 0$ . ■

So at the unique crossing point for the payout and investment thresholds, the latter cuts the former from below; for  $K > 0$ ;  $K_{ID}$ ,  $X_I(K) < X_D(K)$  - investment is the preferred action, and for  $K > K_{ID}$ ;  $1$ ,  $X_D(K) < X_I(K)$  - payouts are the preferred action. This is illustrated in Figure (3).

### 3.6 The Value Function in the Continuation Region.

The firm's value function over the continuation region follows the general form of equation (15). Its precise form is determined by whether  $X_D \geq X_I$  at any particular value of  $K$ . The two solutions are tied together at  $K = K_{ID}$  by the matching conditions at the joint action point  $X_{ID}(K_{ID})$ . The analysis depends crucially on whether the endogenous boundary to the continuation region

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is the investment threshold or the payout threshold. That is whether, for given  $K \in [0; 1]$ ,  $X_D \leq X_I$ . Note that once capacity is fixed at any level it can only stay unchanged or grow. This growth occurs intermittently at the points in time at which investment occurs. With capacity constant, the firm moves up and down a vertical line  $(K; X) \in (0; \min\{X_I(K); X_D(K)\})$  as  $X(t)$  evolves according to the evolution equation (3). First the case  $X_D < X_I$  is considered.

### 3.6.1 The Value function When Payouts Would be Chosen.

Using equation (18) substitute for  $X_D$  in the smooth pasting condition (11) and rearrange to give the following expression for the constant of integration,  $A_1(K)$ ,

$$A_1(K) = \frac{\mu_{-1}(K)}{-\frac{1}{2}(K)} \frac{e^{\frac{-\frac{1}{2}(K)}{\mu_{-1}(K) - \frac{1}{2}(K)}}}{e^{-\frac{1}{2}(K)} e^{\frac{-\frac{1}{2}(K)}{\mu_{-1}(K) - \frac{1}{2}(K)}}} > 0; \quad (27)$$

For a concave production function  $A_1(K)$  is also concave in  $K$ . Substituting equation (27) in equation (17) gives an analytic expression for the value function:

$$V(K; X) = \frac{\mu_{-1}(K)}{-\frac{1}{2}(K)} \frac{e^{\frac{-\frac{1}{2}(K)}{\mu_{-1}(K) - \frac{1}{2}(K)}}}{e^{-\frac{1}{2}(K)} e^{\frac{-\frac{1}{2}(K)}{\mu_{-1}(K) - \frac{1}{2}(K)}}} e^{-\frac{1}{2}(K)X} e^{-\frac{1}{2}(K)X}; \quad (28)$$

$8(K; X) : K \in [K_{ID}; 1]; 0 \leq X < X_D(K) < X_I(K)$ . The typical shape of the value function  $V(K; X)$ , and its components, as a function of  $X$  is displayed in Figure (2).<sup>14</sup> The dashed line is the complementary function in the positive characteristic root. The term below the  $K$ -axis is that in the negative characteristic root. The value function (the sum of these two) lies in between them and is always non-negative.  $V(K; X)$  is defined in this way for  $K \in [K_{ID}; 1], X \in (0; X_D(K))$ ; the baseline parameter case is illustrated in Figure (4).

### 3.6.2 The Value Function when Investment Would be Chosen.

The boundary conditions (4) and (5) form a system of (nonlinear) equations allowing both the investment threshold and the constant of integration to be determined implicitly. Equation (6) forms a linear homogeneous first order ODE for the constant of integration  $A_1$  in  $K$  of the form  $A_1'(K) + m(K) A_1(K) = 0$ . This is reproduced below

$$A_1'(K) + \frac{\frac{1}{2}(K) X_I e^{-\frac{1}{2}(K)X_I} e^{-\frac{1}{2}(K)X_I} - \frac{1}{2}(K) X_I e^{-\frac{1}{2}(K)X_I} e^{-\frac{1}{2}(K)X_I}}{e^{-\frac{1}{2}(K)X_I} e^{-\frac{1}{2}(K)X_I} - e^{-\frac{1}{2}(K)X_I} e^{-\frac{1}{2}(K)X_I}} A_1(K) = 0 \quad (29)$$

<sup>14</sup> The parameter values used in this exercise are  $\frac{1}{2} = 0.1$ ,  $\frac{1}{4} = 0.5$ ,  $\frac{1}{8} = 0.7$ ,  $P = 1$  and  $K = 500$ .

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Equation (29) holds over the region  $(K; X) \in (0; K_{ID_1}] \times (0; X_I(K)]$  subject to,

$$A_1(K_{ID_1}) = \frac{\mu_1(K_{ID_1})}{\mu_2(K_{ID_1})} \frac{e^{-\frac{1}{2}(K_{ID_1})}}{e^{-\frac{1}{2}(K_{ID_1})} e^{-\frac{1}{2}(K_{ID_1})}}; \quad (30)$$

which ensures continuity of the value function at the joint action point. Equation (29) can be solved directly by multiplying throughout by a suitable integrating factor and integrating out. The integrating factor is defined as  $\tau(K) = \exp \int_0^K m(K) dK$ . Simplifying  $m(K)$  in (29), this can be expressed as

$$\tau(K) = \exp \left[ \int_0^K \left( \frac{1}{K} + \frac{1}{e^{AX_I(K)} K^{1-\alpha}} \right) dK \right]; \quad (31)$$

The solution  $A_1(K)$  is then given as  $\frac{C}{\tau(K)}$ , where  $C$  is a constant determined by condition (30).

This allows  $V(K; X)$  to be determined implicitly over the investment threshold as

$$V(K; X) = A_1(K) e^{-\frac{1}{2}X} e^{-\frac{1}{2}X};$$

This is computed directly. First, choose  $\alpha, \beta, \gamma$  and solve for  $K_{ID_1}$  using equation (23) and  $X_I(K)$  using equation (21). Then obtain values for  $A_1(K)$  by solving equation (29) subject to the initial condition (31). This is achieved using a Runge Kutta scheme adapted from Press et al. (1992). The resulting value function holds over the range  $K \in (0; K_{ID_1}), X \in (0; X_D(K)); V(K; X)$  is illustrated for the baseline parameter value case in Figure (4).

### 3.7 Behaviour Outwith the Continuation Region.

To complete the solution to the firm's problem it is necessary to consider behaviour outside the continuation region. For instance, recalling that the investment threshold occurs at strictly positive  $K$  (Proposition 2 b), imagine that a firm initially has no capacity,  $K = 0$ , but high cash balances  $X > 0$ . It could potentially be better off by investing a fraction (to improve its future dividend paying capacity) and so increase shareholder value above  $X$ .

In general, outside the continuation region the firm solves the problem:

$$V(K; X) = \max_{\phi X, \phi K} [\phi X + V(X - \phi X, K - \phi K; K + \phi K)];$$

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where  $(K; X)$  are outside the continuation,  $\Phi X$  is the amount of dividends paid,  $\Phi K$  is the amount of investment undertaken and the value function on the right hand side is on either of the action thresholds. Let  $K_{ID}$  denote the scale of operation intersection between the investment and payout barriers, then if  $K > K_{ID}$  payout is optimal and this brings  $(K; X)$  immediately to  $(K; X - \Phi X)$  which is the payout barrier, while if  $K < K_{ID}$  the optimal policy could be invest only, or investment plus payout depending on the value of  $X$ . The complete description of all the actions the firm can take is illustrated in Figure (3).

### 4 Interpretation and Implications of the Model.

The main results of the model, the mechanisms underlying these results are considered in this section along with implications arising for empirical work.

The main theoretical results concern the separation and nonlinearity of the action thresholds for investment and dividend activity. In particular, for  $K < K_{ID}$ , investment is the preferred action, whenever activity occurs; for  $K = K_{ID}$  the firm is indifferent between investment and dividends so that if action occurs investment and dividend payments occur jointly; finally if  $K > K_{ID}$  dividend payment is the preferred activity should activity occur. This clarifies the role of expansion in determining when it is optimal to switch regimes and is consistent with a life cycle interpretation of firm behaviour, in which young, small firms prefer to take advantage of growth opportunities with the prospect of higher future dividends, while mature, financially secure, firms pay dividends.

These results explain both the endogenous hierarchy different regimes (combinations of investment, payout and inaction) which characterise the behaviour of the firm in different regions of state space and the rules governing transitions between those regimes. This hierarchy is based on the marginal costs of these actions in relation to each other and reflects the price of capital equipment, the value of dividends and the marginal valuation of cash balances and of capital. In this respect the optimal dividend and investment decision modelled here bear some resemblance to the hierarchy of activities observed in models of multifactor demand in the presence of non-convex adjustment costs, see Eberly and Van Mieghem (1997).

These results appear to depend ultimately on the concavity of the revenue function in capital

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stock, and on the complementarity of the irreversibility and financial constraints. This complementarity can be straightforwardly understood. If a firm faces symmetric convex costs of adjusting capital stocks then it can manage its physical capital stocks prudently to ward off the threat of liquidation, which is what happens in Milne and Robertson's (1996) model. If capital investment is irreversible, such prudent management is not possible once capital is in place, leading to greater caution in the investment decision.<sup>15</sup> The juxtaposition of irreversibility and financial constraints will have a greater impact on investment activity than either constraint individually. In addition, financial constraints of the form considered here may be more prevalent when the firm undertakes irreversible investments. This is because a firm is much less likely to be able to raise external finance if the projects it undertakes require irreversible investment expenditures, since potential providers of external finance will be less willing to supply funds when the project is irreversible.

The model has implications for empirical work. At a bare minimum, the possibility raised here, that irreversibility and financial constraints may bind simultaneously, suggests that existing work needs to be reassessed in order to filter out the impact of "the other" constraint and correctly assess the impact of the constraint in question on firm behaviour.<sup>16</sup> Otherwise, since both constraints tend to reduce the incentive to invest, the magnitudes of the effect of each "pure" constraint may be biased upwards, leading to incorrect policy recommendations. In tests of the impact of financial constraints on investment activity, one should control for the extent of irreversibility.<sup>17</sup>

Similarly, when examining the implications of irreversibility for investment activity, one should control for the presence of finance constraints.<sup>18</sup>

If irreversibility accentuates the effect of financial constraints, then controlling for irreversibility becomes a more important issue in the examination of the impact of financial constraints. This suggests that, to isolate effects of the latter, one should work primarily with data on firms whose

<sup>15</sup>It is likely that this would carry over to models of costly reversibility based on non-convex adjustment technology, as in Abel and Eberly (1996). This is because the non-convexity will lead to a capital loss if resale to the secondary market is undertaken, and this engenders caution in the initial investment decision. The analysis of Section (3) suggests that such an effect is accentuated by financial constraints.

<sup>16</sup>Researchers who are dubious of the relevance of irreversibilities might point to the evidence documented by Hubbard (1998), that linear investment equations (derived from models with convex costs of adjustment) are not rejected for those firms classified as financially unconstrained and argue that non-convex adjustment cannot be an important issue. However, as Hamermesh and Pfann (1996) discuss, space and time aggregation smooths (and linearises) investment observations, even at the level of the firm. This may explain the success of linear investment equations despite the relevance of non-convexities for investment decisions.

<sup>17</sup>A straightforward way to do this is to use data on the difference between primary and secondary market prices for the relevant capital equipment, the author is unaware of any empirical studies which do so.

<sup>18</sup>In the empirical literature on the consequences of irreversible investment, only Guiso and Parigi (1999), to date, appear to have made any effort to control for financial constraints.

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investments are not irreversible, and vice versa.

To illustrate what failure to control for "the other" constraint might mean in practice, consider the following example. One of the tests researchers have devised for the effects of irreversibilities on investment activity is via the sign of the investment uncertainty relationship.<sup>19</sup> In the model in section (3) variations in uncertainty affect the incentive to invest through different channels.<sup>20</sup> First there is a direct effect of irreversible investment (which many researchers expect to be negative if irreversibilities are present). Second there is a direct effect through financial constraints, via the precautionary motive, which induces the firm to display prudence to ward off the threat of liquidation (i.e. prudent behaviour could occur even without irreversibility, as in Milne and Robertson (1996), but would not lead to inaction). Finally there is the indirect effect of irreversibility, through its complementarity with financial constraints, which again reduce the incentive to invest as uncertainty increases. Thus a negative relationship between variation in uncertainty and investment activity could reflect the pure precautionary motive or the interaction effect, and need not reflect the pure irreversibility effect to which it is ascribed in existing studies. This underscores the need to identify and control for different constraints.

Finally, consider what testable implications emerge directly from the model and how one might proceed to examine them. One of the main features of models based on non-convex adjustment technologies is the existence of an inaction region and of threshold-type behaviour. Due to time and space aggregation issues, even trying to observe investment zeroes can be difficult. This appears to make direct tests of the fine structure of the model, such as the level of  $\frac{X}{K}$  at the investment threshold at different scales of operation infeasible, at least using the standard econometric approaches. An alternative approach which exploits the economic structure of the model to get a

measure of the threshold, without estimating a full dynamic structural model, has been developed

<sup>19</sup>Direct tests of the impact of irreversibility are difficult since the main implications of the model are not for investment itself but for the location of the threshold at which investment occurs. Empirical work has focussed on the sign of the relationship between uncertainty and investment, with a negative sign being taken as consistent with the option to wait effect under irreversibility and ongoing uncertainty. See Carruth, Dickerson and Henley (2000) for a survey of early empirical work. Such an approach is problematic since theory suggests both that increases in uncertainty widen the inaction band by increasing the value of the option to wait, and that the increased volatility of the underlying process may raise the probability of investment occurring, see Sarkhar (2000) for a recent treatment. Potentially, irreversibility may lead to a increased uncertainty to reduce investment in the short-run, in that, if uncertainty increases, the shift in the threshold (widening of the inaction region) can lead, at least initially, fewer firms to be in the neighbourhood of the (new) investment threshold.

<sup>20</sup>Sensitivity analysis, omitted from this version of the paper, but available from the author on request, found that the investment and dividend action thresholds of the model in Section (3) both shifted outwards in response to a rise in the level of uncertainty.

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by Pindyck and Solimano (1993). The basic idea is to exploit assumptions (about production technology, market structure etc.) made in developing the theoretical model, in order to construct a measure of the marginal value product of capital for a given unit (over time).<sup>21</sup> The highest values of this measure (over time) give an estimate of the investment threshold for a given unit, one can then examine the impact of variations in uncertainty on the threshold value (across units). To date this approach appears only to have been applied to aggregate and sectoral data, but it could provide a useful approach at the firm level. In the context of the model in Section (3), one would expect to observe that i) the estimated threshold value of the marginal value product of capital and ii) the magnitude of the (negative) relationship between (changes in) uncertainty and investment, would both be higher a) for firms identified as financially constrained (through their dividend policy) than for those which were financially unconstrained, and b) for those firms whose investment were primarily irreversible, as opposed those for which investment are more reversible.<sup>22</sup> This suggests how one might identify behaviour consistent with the model.

The implications of the model suggest a re-appraisal of existing empirical studies is required, but before accepting such a view, it is important to know whether these implications are robust to variations of the economic environment. This is the subject of the next section.

## 5 Generalisations

The results on the separation of the action thresholds, on their curvature and on the endogenously determined hierarchy of actions, established in Section (3), are obtained in the context of a specific restricted model. Naturally it is important to know how robust these results are likely to be once these restrictions are relaxed. The model is extended to allow for depreciation of capital, relaxation of the financial constraint, explicit treatment of demand uncertainty, and non-residual dividend policy. As explained in the last section the main features giving rise to the results are the assumption about the curvature of the revenue function and the complementarity of the two constraints. While each generalisation enriches the model, the basic results remain intact because

<sup>21</sup>This would be the marginal value of assets in place using the terminology of Dixit and Pindyck (1994).  
<sup>22</sup>Note that the implications of the model are consistent with the methodology of empirical work on financial constraints and investment activity, which suggests that dividend policy, size and age, should be used to classify firms as financially constrained, Hubbard (1998).

the concavity and complementarity features are preserved.

## 5.1 Depreciation

Depreciation would enrich the set of feasible sample paths, allowing movement between various regimes which, in the restricted model, were inaccessible once other regimes had been entered. For example, with depreciation, capital stocks decline in the absence of investment, so firms may cycle around the joint action point, allowing joint action to become a more frequent occurrence..

Also, since investment decisions are effectively less reversible once depreciation is admitted, investment may occur at lower  $X$  for given  $K$  (this would be consistent with the literature on irreversible investment without financial constraints), while the increased requirement for internal funds (to replace depreciated capital) may lead the dividend threshold to shift outwards. These effects would lead joint action to occur at a higher  $K$ , but there is no reason to expect that the investment threshold to become linear or convex in  $K$ . The fact that the concavity of the revenue function and the complementarity of the constraints are preserved suggests that the basic separation and curvature properties of the thresholds will be maintained - although it is no longer possible to guarantee that there will only be one joint action point.

## 5.2 Financial Constraints

An important question is whether the results continue to hold on relaxing the restriction that the firm has access only to internal funds and faces liquidation if these fall too low. Clearly if external finance were available at no premium, the firm's problem would return to that of the canonical irreversible investment model. However, suppose that there were an upper bound (low enough to make the problem interesting) on the availability of external funds - this could be incorporated without introducing extra state variables simply by adding these external funds to the available internal funds,  $X$ . This suggests that the problem is little changed since the constraint complementarity and revenue function concavity property properties, which drive the results, remain intact. For such a relaxation to be interesting there must be some reason for a (meaningful) limit to be imposed on the availability of external finance. One good reason for such a



feature is that firms with highly irreversible prospective investments are precisely the sort of firms that lenders would be reluctant to lend to, because of the capital losses that would be endured in case of a bad state of the world. Of course this is another illustration of the complementarity of the irreversibility and financial constraints.<sup>23</sup>

### 5.3 Demand Uncertainty

In the model of Sections (2) and (3), demand uncertainty is captured in reduced form through equation (2), but this makes direct comparison with the results of the literature on irreversible investment difficult. For instance, without financial constraints, under suitable assumptions about the production and demand functions, it is possible to obtain a closed form solution in which the level of demand,  $Z$ , at which investment is triggered is a linear increasing function of scale of operation,  $K$ , whereas the investment threshold in Section (3) determines  $X_I$  as an increasing concave function of  $K$ . It would be interesting to know whether either or both of these features is preserved if demand uncertainty is introduced directly.

Incorporating demand uncertainty directly (perhaps with the evolution of demand modelled as a geometric Brownian motion) introduces a number of interesting possibilities. For instance a firm with high demand, high internal funds and low capital stock will have an incentive to invest to take advantage of high demand opportunities.<sup>24</sup> On the other hand a firm with low demand, high internal funds and high capital stock, recognising its low growth opportunities, may be inclined to payout. The presence of high demand (and profits) could allow the firm to be less concerned about the threat of liquidation. These possibilities give some insight into the extra flexibility generated by modelling demand uncertainty directly. On the other hand the flexibility comes at the cost of analytical tractability - both the investment and payout thresholds will be surfaces in  $(K; X; Z)$ -space, and there is no clear guide as to how they intersect. Nonetheless, the constraint complementarity and concavity features that generated the results in Section (3) still hold, so there is certainly no reason to expect the investment and dividend thresholds to be superposed.

<sup>23</sup>If there were no upper bound on the availability of external finance, but the latter required a premium over internal finance, then the equivalence to the problem in Section (3) may not follow. Nonetheless, the complementarity of the constraints would still be important to the extent that firms undertaking more irreversible investments would presumably face a higher premium on external finance than firms whose undertaking more reversible.  
<sup>24</sup>Presumably the incentive to invest will be attenuated if the process governing demand is mean reverting rather than exhibiting the persistence of a geometric Brownian motion.

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Also, the investment threshold,  $X_I(K)$ , holding demand,  $Z$ , constant may well continue to vary nonlinearly in scale of operation. Moreover as demand increases, this investment threshold  $X_I(K)$  could be expected fall at each value of  $K$ ; as higher demand increases the incentive to invest. From a different perspective, holding internal funds,  $X$ , constant, the level of demand which triggers investment,  $Z_I(K)$  in the financially constrained case would be expected to lie above that which triggers investment in the financially unconstrained case in  $(K; Z)$ -space, and as the level of internal funds rises, the investment threshold would be expected to converge towards that in the financially unconstrained case.<sup>25</sup> In short, adding demand uncertainty enriches the problem, but does not remove the key features that drive the results.

### 5.4 Non-Residual Dividend Policy

Some readers might take exception to the characterisation of dividend policy as a residual in firm's decisions.<sup>26</sup> Although dividend policy can be argued to be consistent with a life-cycle view of the firm, there are other features of observed payout behaviour that it would be interesting to capture. In particular, firms appear to smooth dividends, and dividends appear to have some information content for equity-holders. One popular explanation for this is that dividends play a signalling role conveying information about the level and variability of expected earnings, Miller and Rock (1985). In particular, dividend initiations and increases convey good news to the relatively uninformed equity-holders, while dividend omissions and reductions convey bad news. Moreover, there is some evidence that dividend initiations/increases convey information regarding the lower risk of the firm, Dyl and Weigand (1998). Might not a non-residual dividend policy alters the nature of the investment threshold obtained in Section (3) ?

To begin to capture such phenomena, one could consider the dividend initiation decision as irreversible. That is, the firm either pays no dividends, or commits irreversibly to pay an exogenous fraction  $\alpha \in (0; 1)$  of expected earnings. In return for this commitment the firm, by signalling lower risk, can offer a lower rate of return,  $\gamma_1 \in (0; \gamma_0)$  where  $\gamma_1$  is the required rate of return which

<sup>25</sup> This could be checked by obtaining the numerical solution to equivalent problems for a financially constrained firm and unconstrained firm, and plotting the investment threshold (level of demand as a function of scale of operation), both for the financially unconstrained case and at a variety of different levels of internal funds for the financially constrained case.

<sup>26</sup> In fact this approach is used in the empirical work on the role of financial constraints in investment activity, in which dividend policy is used to identify those firms facing financial constraints, Hubbard (1998), pp202-203.

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applies prior to dividend initiation.. So in place of equation (5)

$$D(t) = \begin{cases} 0; & \frac{1}{2} = \frac{1}{2}_0 \\ K^0; & \frac{1}{2} = \frac{1}{2}_1 \end{cases}$$

while the other equations (2), (3), (4), (6) and (7) remain as before.<sup>27</sup>

Signalling models distinguish between the informed firm and its uninformed owners, this can be handled by rewriting the problem from the viewpoint of the firm, which now maximises the discounted present value of expected future profits, where in Sections (2) and (3), it maximised the discounted present value of expected future dividends.

The dividend initiation aspect of this problem can be solved in a similar fashion to an irreversible start up decision for a discrete project (of the form discussed in Dixit and Pindyck (1994), Chapter 5). That is, first determine the form of the solution for  $D = 0$  (call the value function  $V^0$ ). Next determine the form of the solution for  $D = K^0 > 0$  (call the value function  $V^1$ ). Finally, determine the optimal dividend initiation point using value matching and smooth pasting conditions ( $V^0 = V^1$ ,  $V_X^0 = V_X^1$ ) to tie the solutions together.<sup>28</sup> The twist is that within the  $D = 0$  and  $D > 0$  regions the firm solves an incremental irreversible investment problem with boundary conditions given by equation (12). The fact that the constraint complementarity and revenue function concavity properties still hold, that value function takes a similar form (with the addition of a particular solution of the form  $K^0 = \frac{1}{2}i$ , where  $i \in [0, 1]$ ) and that the boundary conditions are identical to those in Sections (2) and (3) strongly suggests that the investment threshold will be concave and increasing in  $K$  as in equation (19).

One unresolved issue is whether, at a given scale of operation, the firm would be more or less willing to invest when  $D > 0$  as opposed to when  $D = 0$ . This is the same as asking whether in  $(K; X)$ -space the investment threshold when  $D > 0$  lies above the equivalent threshold when  $D = 0$ . There are two effects to take into account. First, when  $D > 0$ , the required rate of return

is lower:  $\frac{1}{2}_1 < \frac{1}{2}_0$ . This would give the firm a greater incentive to invest, and tend to reduce the

<sup>27</sup>Note that, once dividends are initiated, they are much smoother than realised net earnings and may increase with scale of operation. Of course  $\frac{1}{2}$  should be determined endogenously, but this complication is ignored for simplicity.

<sup>28</sup>Note that the firm doesn't incur a sunk cost to initiate dividends, it undertakes to pay the dividend. For this reason such a model is a rather weak characterisation of this commitment. The firm is committed to paying non-zero dividends, even in the face of liquidation, because we assume it so. Certainly such a model offers only a loose resemblance to a signalling model. Nonetheless, this conveys the favour of a non-residual based dividend policy and allows discussion of the impact of a non-residual dividend policy on investment activity.

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investment trigger value,  $X_I(K) \geq K$ , once the firm starts to pay dividends (compared with when it pays no dividends). Against this dividend initiation commits a fraction of expected profits to a particular use, which increases the threat of liquidation, and will tend to make the firm more wary of committing to irreversible investment. Commitment tends to raise the trigger value,  $X_I(K) \geq K$ , when the firm pays dividends compared with when it does not. The magnitude of these effects depends on the particular parameter values, but suppose for expositional purposes that the latter effect dominates, so that the investment threshold when dividends are paid lies above that which obtains prior to the initiation of dividends.

In general, at each level of capacity,  $K$ , there is some  $X = X_I(K)$  at which investment are triggered, and some  $X = X_D(K)$  at which dividends are initiated. In the  $D = 0$  region, these thresholds are such that  $X_D(K) > X_I(K)$  and the firm solves the incremental investment problem for  $D = 0$ . Since, internal funds can grow more quickly when the firm is not committed to distributing a fraction of its earnings, there will, in the absence of liquidation, come a point at which internal funds are high enough, and the marginal value product of capital low enough, that the firm can initiate dividend payments. At this point  $X_D(K) = X_I(K) = X_{ID}(K_{ID})$ . Following dividend initiation the firm solves the incremental investment problem with  $D > 0$ , so there may be a discontinuity in the operational investment threshold at the joint action point.

Unfortunately, it is less easy to obtain analytic results on the joint action point for the dividend initiation problem, because one has to use information on two implicit functions (one defining each investment threshold) and the matching and smooth pasting conditions for dividend initiation. By contrast in Section (3) the closed form for the dividend threshold facilitates analytic results.

The model shows that, even in the face of a non-residual dividend policy the investment threshold will continue to be a concave increasing function of  $K$  - although it may now exhibit a discontinuity around the dividend initiation point. Dividend policy will continue to have a life-cycle interpretation. The robustness of these features of the model to variations in the structure of dividend policy once again stems from the complementarity of the irreversibility and financial constraints and the concavity of revenues in  $K$ .

## 6 Conclusion.

This paper has considered investment and dividend policy under the combined effect of financial constraints and irreversibility. Given the growing empirical evidence for the pervasiveness of each of these constraints separately the current paper takes the obvious next step and provides a unified theoretical analysis of the dynamic consequences of the constraints and should provide a point of departure for future empirical work. The main lesson to draw from this paper concerns the nature of the interdependency of investment and dividend policy under irreversibility and financial constraints. Complementarity of these constraints has important consequences for empirical work.

The key theoretical results are those of the separation of the thresholds and the nonlinearity of the threshold in  $K$ . These arise directly from the concavity of the production function in  $K$  in conjunction with the complementarity of the irreversibility and financial constraints. These features ultimately generate the hierarchy of activities thresholds which are endogenously determined, conditional on the location of the firm within state space, on the basis of the costs of each of the activities. As a result of complementarity and concavity of the production function, the optimal policy depends on the scale of operation in manner consistent with a life cycle view of the firm in which small (young) firms with good growth opportunities (high marginal value product of capital) retain funds to expand, while large, secure (mature) firms, those with low marginal value product of capital and low marginal value of internal funds choose to pay dividends.

The model has some important implications for empirical work. First the mere possibility of the joint presence of irreversibility and financial constraints, which is basically ignored in the empirical literature, indicates that the magnitude of the effects ascribed to a single constraint may be mismeasured, since both constraints would affect the incentive to invest in a similar manner. The complementarity of the two constraints suggests that separating out the "pure" effect of a single constraint is best done in the case of financial constraints by working with data in which investments are as reversible as possible, and in the case of irreversibilities, by working with firms that are financially unconstrained.

The analytic results on the separation of the thresholds were derived under specific functional

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forms and strong versions of irreversibility and financial constraints. However, the general properties of threshold separation and nonlinearity of the thresholds in  $K$  seem likely to be robust to a range of more complex environments, such as allowing for capital depreciation, relaxing the financial constraint and explicit consideration of demand uncertainty. The robustness seems to follow precisely because in all the generalisation considered, complementarity of the constraints and concavity of the production function are preserved.

The results suggest a number of directions for future work. First, as discussed in Section (4) empirical work on investment should proceed by allowing and controlling for both irreversibility and financial constraints and existing results should be reassessed. Second, the generalisations suggested in Section (5) (and others) could be examined more formally using numerical methods. Third the nature and consequences of the constraint complementarity effect for lenders and borrowers decisions could be examined using an asymmetric information approach to obtain a different perspective.

## Appendix A

### A Appendix.

This Appendix contains proofs of various statements in the text.

#### A.1 Proof of Propositions 2.

Proof. a) Since the costs of control are linear in the extent of dividend payment, the firm's optimal dividend policy will take a bang-bang form, (instantaneous barrier control). Dividend policy will be either to pay no dividends or to pay at the maximum possible rate. Since the value function is increasing and concave in  $K$  and  $X$  there will be a single (continuation) region over which it is optimal to retain earnings and pay no dividends. The boundary of this region will be characterised by some threshold level of cash balances,  $X_D$ , such that for  $X > X_D$  it is optimal for the firm to distribute cash balances as dividends. This is because, given the curvature of the value function, the marginal value of cash balances falls as  $X$  rises; above the threshold level,  $X > X_D$ , the marginal value of cash in hand is less than its value paid out to shareholders. Unless  $V_{KX}(X_D; K) = 0; \forall K$ , the payout threshold,  $X_D$  is a function of capacity  $X_D = X_D(K)$ .

Next, consider the bounds on the location of the trigger threshold in the payout region of state-space. Suppose the firm sets  $X_D = 1$ . Then dividends are not paid out before the end of the planning horizon. In that case, since  $0 < \frac{1}{2}$ , the value of the firm to the shareholders is  $V(K; X) = \lim_{T \rightarrow \infty} \int_0^T e^{-\rho s} K(s) ds = 0$ . Therefore the firm must always do better by setting  $X_D < 1$ , provided it has strictly positive cash holdings and capacity. Suppose instead that the firm chooses  $X_D = 0$ , then it enters liquidation immediately. It gains nothing from its existing non-negative holdings of physical capital since these have zero resale value, nor is it compensated for the earnings associated with this capacity lost through liquidation. Therefore a dividend policy with  $X_D = 0$  is sub-optimal. From Conjecture (1), since  $V_X(K; 0) > 1$ , the initial increment to retained earnings,  $X$ , is more valuable than distribution of earnings, therefore the firm chooses  $X_D \in (0; 1)$ . This establishes statement a):

b) The linearity of the value function in the control variables ensures that the firm's investment policy is either to invest nothing or invest at the maximum possible rate (instantaneous barrier

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control). Since the firm's value function is continuous and concave in capacity and cash balances throughout the continuation region, there will be a single region in  $(K; X)$  space over which inaction is the optimal investment policy and a single region over which it is optimal to invest, see Dixit and Pindyck (1994). The boundary between the two regions, the threshold  $K_I$ , will be such that for  $K < K_I$ , it is optimal for the firm to invest. This follows from the continuity and curvature of the value function. Investment reduces the marginal value product of capital, so it is only worthwhile investing when capacity is low enough that the value of the marginal unit of cash in hand invested in capacity exceeds the value of retaining earnings.

Next, consider the bounds on the investment trigger. Suppose  $K_I < K(X_I) = 1$ , then, assuming  $P$  strictly non-negative, the firm inevitably goes into liquidation trying to meet the cost of the investment expenditures involved in attaining the desired threshold level of capital stock. Clearly, this is not optimal since, on liquidation, the secondhand value of the installed physical capital is zero. Instead, suppose that  $K_I < K(X_I) = 0$ . Then the firm's value is bounded below by the value of its outstanding cash in hand  $V(0; X) \geq X$ . Continued operation yields no profit, since  $K = 0; \delta t \geq 0$ . In this case, given shareholders' impatience, the firm's optimal dividend policy is immediate distribution. Again, as a consequence of immediate payout, this policy is uninteresting. Provided  $V_K(0; X) > P$ , given the continuity and curvature of the value function, it follows that  $K_I \in (0; 1)$ . This establishes statement b). ■

### A.2 Proof of Proposition 3.

Proof. First impose the boundary conditions at the trigger point  $X_D$ .

$$V_X(K; X_D) = A_1(K) \frac{h}{\gamma_1(K)} e^{-\gamma_1(K)X_D} + \frac{i}{\gamma_2(K)} e^{-\gamma_2(K)X_D} = 1 \quad (\text{A.1})$$

$$V_{XX}(K; X_D) = A_1(K) \frac{h}{\gamma_1(K)} \gamma_1(K)^2 e^{-\gamma_1(K)X_D} + \frac{i}{\gamma_2(K)} \gamma_2(K)^2 e^{-\gamma_2(K)X_D} = 0 \quad (\text{A.2})$$

Rearranging (A.2) gives a closed form expression for  $X_D$  as a function of the characteristic roots,

$$X_D(\gamma_1; \gamma_2) = \frac{\ln \frac{\mu_3 \gamma_2^2 \gamma_1}{\gamma_1 \gamma_2}}{\gamma_1 - \gamma_2} > 0: \quad (\text{A.3})$$



[illegible]

Again, suppose  $K/X \rightarrow 1$  such that  $\frac{X}{K} = -\alpha$ , an arbitrary positive constant. Then  $\frac{X}{K} = \frac{X}{K} K^{1/\alpha} = -\alpha K^{1/\alpha}$ . Clearly  $\lim_{K/X \rightarrow 1} \frac{1}{K} \frac{X}{K} = -\alpha \frac{X}{K} = 1$ , therefore, the first exponential term in equation (20) becomes unbounded, whilst the other terms tend to zero (at least geometrically) in

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the limit. So  $\lim_{K;X \rightarrow 1} G(K;X) = 1$ . The final asymptotic property of  $G(K;X)$  to be considered here is the limiting behaviour as  $K;X \rightarrow 0$ , such that  $\frac{X}{K} = -1$ , an arbitrary positive constant. The elements of equations (20) containing the exponential terms  $e^{A \times K^{i^*}}$  and  $e^{i A \times K^{i^*}}$  are then constant and  $G(K;X)$  can be written as

$$G(K;X) = \frac{6}{4} \left( C_1 e^{A-1} K^{i^*+1} + C_2 e^{i A-1} K^{i^*+1} + \frac{7}{5} \right) + \frac{3}{5} \left( A^2 P K^{i^*+2} + A^2 - 1 K^{i^*+1} + C_3 K^{i^*+1} \right)$$

In the limit as  $K;X \rightarrow 0$ , the dominant term is that in  $K^{i^*+2}$ , which is uniformly positive. It follows that  $\lim_{K;X \rightarrow 0} G(K;X) = 0^+$ .

The foregoing analytic arguments provide some insight into the asymptotic properties of the function  $G(K;X)$ , and therefore into its overall shape. The arguments are also consistent with the particular parameterisation used in Figure (1).

**Proposition 7**  $\delta K > 0$ , a solution,  $X_1(K)$ , to the equation  $G(K;X) = 0$  exists, where  $G(K;X)$  is defined by equation (20).

**Proof.** Since  $G(K;X)$  is clearly continuous in  $K$  and  $X$  on  $(K;X) \in (0;1) \times (0;1)$ , it follows from lemma (6) that a solution to  $G(K;X) = 0$  exists.<sup>29</sup> ■

<sup>29</sup> This analysis does not establish whether a solution to the equation  $G(K;X) = 0$  is unique. Uniqueness cannot be addressed using numerical analysis of the solution for particular parameter values, since one can only establish uniqueness within a finite region of  $(K;X)$ -space, for particular (finite) parameter values, global results can not be obtained using these techniques, and to proceed implicit expressions for  $X_1(K)$  and  $A_1(K)$  must be found.

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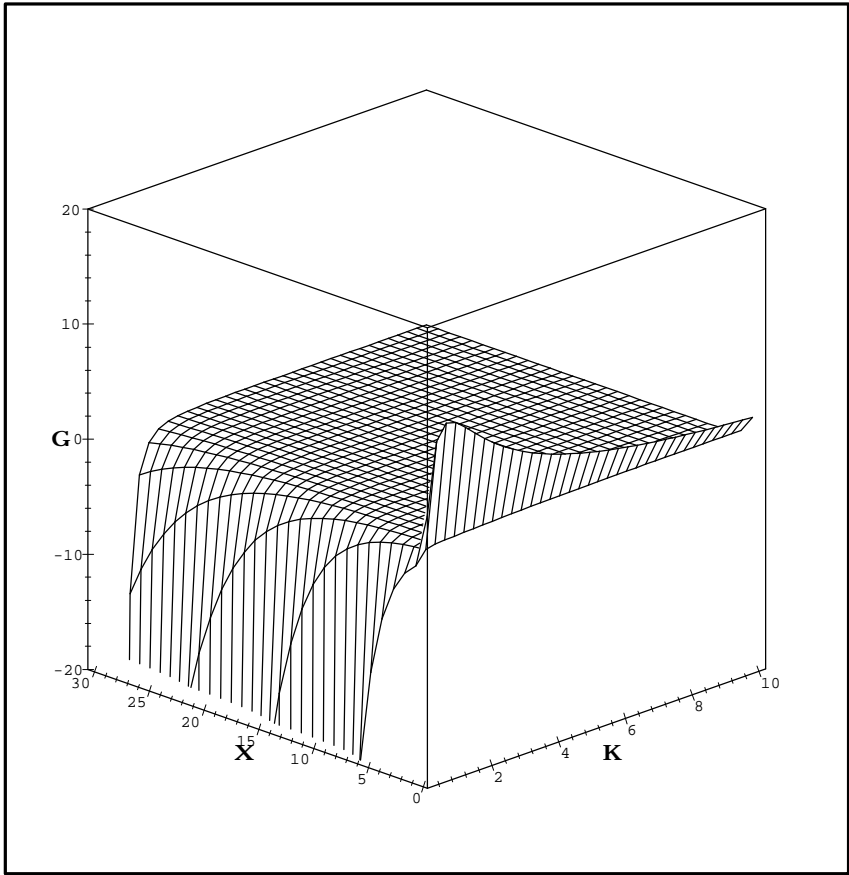


Figure 1: The  $G(K; X)$  Surface.

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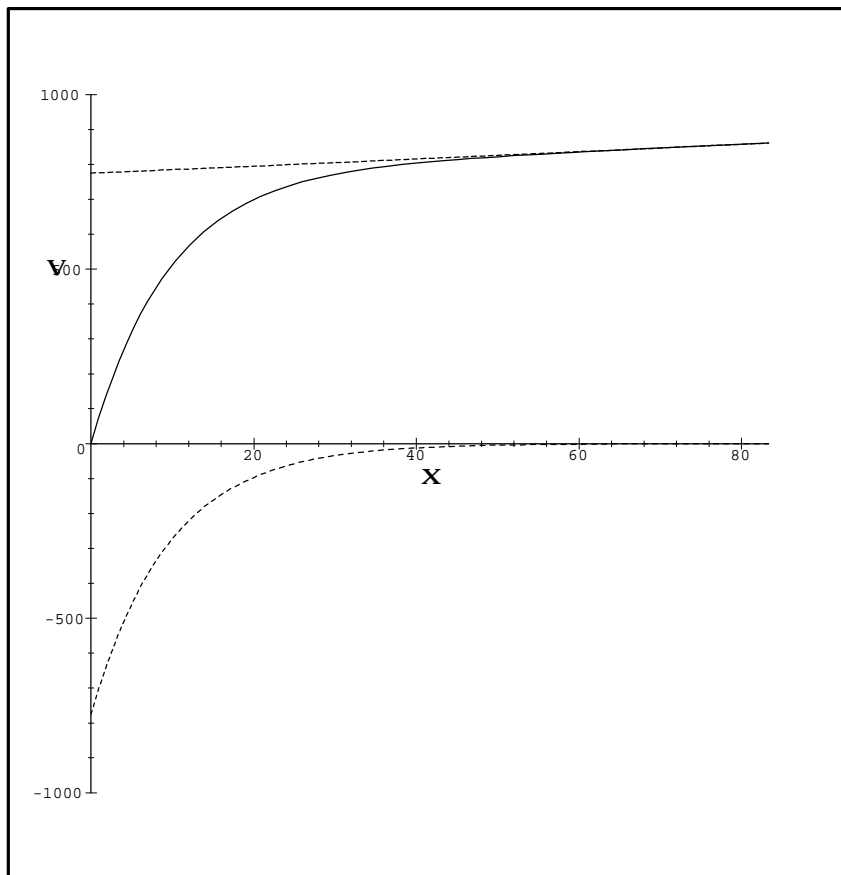


Figure 2: The Value Function,  $V$ , and its Components as Functions of  $X$ .

INVESTMENT AND DIVIDENDS UNDER IRREVERSIBILITY AND FINANCIAL  
CONSTRAINTS

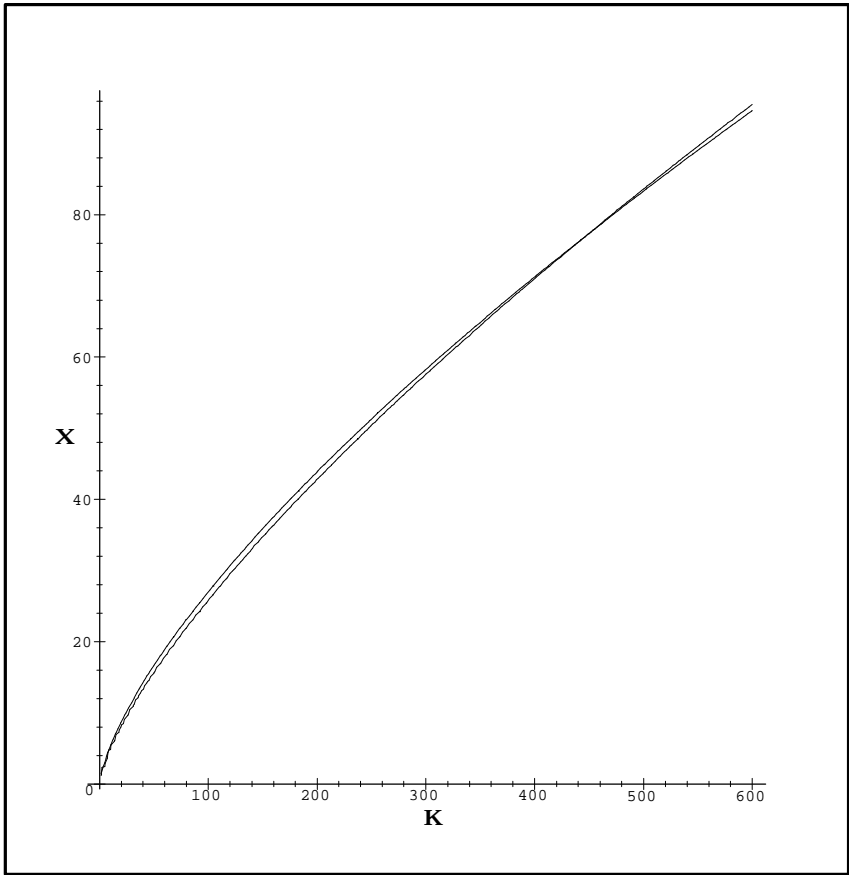


Figure 3: Intersection of the Dividend and Investment Thresholds.

## INVESTMENT AND DIVIDENDS UNDER IRREVERSIBILITY AND FINANCIAL CONSTRAINTS

Figure 4: Value Function in Dividend and Payout Regions.